3) Initial deformations of relatively small magnitude can have significant effects on stresses and displacements of loaded structures. Consequently, unspecified or inadvertent initial deformations may reduce the reliability of structures optimized using procedures that do not adequately account for the effects of such deformations.

Appendix: Derivatives of Stresses and Displacements for Prestressed Trusses

The stiffness relations between applied forces and developed displacements for a linear elastic structure are of the form

$$[K]\ \{u\}=\{F\}=\{F^m\}+\{F^0\} \tag{A1}$$

where $[K]$ is the stiffness matrix, $\{F\}$ the vector of equivalent nodal loads and $\{u\}$, the vector of nodal displacements. The forces are the sum of the applied mechanical loads $\{F^m\}$ and the initial prestress loads $\{F^0\}$, where expressions for $\{F^0\}$ are given in numerous references.[11,12]

Differentiating Eq. (A1) with respect to design variable d_j gives, for load condition ℓ,

$$[K]\frac{\partial\{u\}_\ell}{\partial d_j}+\frac{\partial[k]}{\partial d_j}\{u\}_\ell=\frac{\partial\{F^m\}_\ell}{\partial d_j}+\frac{\partial\{F^0\}}{\partial d_j} \tag{A2}$$

The mechanical loads are independent of both the area design variables and the prestress design variables; consequently, $\partial\{F^m\}_\ell/\partial d_j=0$.

The element areas A_i are linked to the area design variables d_n through a linking matrix with components a_{in}

$$A_i=a_{in}d_n \quad n=1,2,\ldots,N \quad i=1,2,\ldots,I \tag{A3}$$

The gradient of the prestress forces with respect to area design variables is, for truss elements

$$\frac{\partial\{F^0\}}{\partial d_n}=\sum_{i\epsilon n}a_{in}\frac{\partial\{F^0\}}{\partial A_i}=\sum_{i\epsilon n}a_{in}Eu_i^0\{B\}_i \tag{A4}$$

where E is the elastic modulus, u_i^0 the initial displacement, and $\{B\}_i$ the linear strain-displacement vector.

The gradient of the displacement with respect to the area design variables is obtained by substituting Eq. (A4) into Eq. (A2) and solving for $\partial\{u\}_\ell/\partial d_n$; this gives

$$\frac{\partial\{u\}_\ell}{\partial d_n}=[K]^{-1}\left(\sum_{i\epsilon n}a_{in}Eu_i^0\{B\}_i-\frac{\partial[K]}{\partial d_n}\{u\}_\ell\right) \tag{A5}$$

where the stiffness gradients $\partial[K]/\partial d_n$ are constant and need to be calculated only once.

The member initial displacements u_i^0 are assumed to be linked to the prestress design variables d_m through a linking matrix with components $\hat{a}_{i,m-N}$,

$$u_i^0=\hat{a}_{i,m-N}d_m \qquad m=N+1,\ldots,M \tag{A6}$$

where M and N are integers defined previously.

The gradient of the prestress force with respect to the prestress design variables is

$$\frac{\partial\{F^0\}}{\partial d_m}=\sum_{i\epsilon m}\hat{a}_{i,m-N}\frac{\partial\{F^0\}}{\partial u_i^0}=\sum_{i\epsilon m}\hat{a}_{i,m-N}EA_i\{B\}_i \tag{A7}$$

The gradient of the displacements with respect to the prestress design variables is, from Eqs. (A2) and (A7)

$$\frac{\partial\{u\}_\ell}{\partial d_m}=[K]^{-1}\sum_{i\epsilon m}\hat{a}_{i,m-N}EA_i\{B\}_i \tag{A8}$$

where the gradient of the stiffness matrix with respect to the prestress variables has been taken as identically zero.

The stress in member i under load condition ℓ is

$$\sigma_{i\ell}=E(\epsilon_{i\ell}-\epsilon_{i\ell}^0)=E\left(\{B\}_i^T\{u\}_\ell-\frac{u_i^0}{L_i}\right) \tag{A9}$$

where L_i is the length of the member.

The gradient of the stress with respect to the area design variables is

$$\frac{\partial\sigma_{i\ell}}{\partial d_n}=E\left(\{B\}_i^T\frac{\partial\{u\}_\ell}{\partial d_n}-\frac{1}{L_i}\frac{\partial u_i^0}{\partial d_n}\right)=E\{B\}_i^T\frac{\partial\{u\}_\ell}{\partial d_n} \tag{A10}$$

where the displacement gradient $\partial\{u\}_\ell/\partial d_n$ is given by Eq. (A5). The gradient of the stress with respect to the prestress design variables is

$$\frac{\partial\sigma_{i\ell}}{\partial d_m}=E\left(\{B\}_i^T\frac{\partial\{u_\ell\}}{\partial d_m}-\frac{1}{L_i}\hat{a}_{i,m-N}\right) \tag{A11}$$

where the displacement gradient $\partial\{u\}_\ell/\partial d_m$ is given by Eq. (A8).

Acknowledgment

This research was supported by AFOSR Grant No. 74-2640A; W. Walker, Program Manager, L.A. Schmit, Jr., R.B. Nelson, and L.P. Felton coprincipal investigators.

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Eigenrelations in Structural Dynamics

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Introduction

THE second-order system of ordinary differential equations representing free vibrations for a system with n degrees of freedom

$$[M]\ \{\ddot{x}\}+[C]\ \{\dot{x}\}+[K]\ \{x\}=\{0\} \tag{1}$$

Received Jan. 24, 1977; revision received March 29, 1977.
Index categories: Vibration; Structural Dynamics.

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corresponds to the eigenvalue problem

$$\left[[M]\ \lambda_j^2+[C]\ \lambda_j+[K]\right]\ \{u_j\}=\{0\} \tag{2}$$

in which $[M]$, $[C]$, and $[K]$ are the mass, damping, and stiffness matrices, respectively; $\{u_j\}$ is the jth eigenvector associated with the jth eigenvalue, λ_j. The system is linear with constant coefficients, and the matrices are symmetrical with either the mass and/or stiffness matrix being positive definite. The matrices usually are banded and large.

A state space representation of Eq. (1) is

$$[A]\ \{y\}-[B]\ \{\dot{y}\}=\{0\} \tag{3}$$

in which

$$\{y\}=\left\{\begin{matrix}\{x\}\\ \{\dot{x}\}\end{matrix}\right\} \tag{4}$$

$$[A]=\begin{bmatrix}[K] & [0]\\ [0] & -[M]\end{bmatrix} \tag{5}$$

$$[B]=\begin{bmatrix}-[C] & -[M]\\ -[M] & [0]\end{bmatrix} \tag{6}$$

and the eigenvalues and eigenvectors satisfy

$$\left[\ [A]-\Lambda_j[B]\ \right]\ \{v_j\}=\{0\} \tag{7}$$

Equations (4-7) do not represent a unique representation of the state space equations, but are chosen here to maintain symmetry in the matrices $[A]$ and $[B]$.

In this Note, the relations between the eigenvalues and eigenvectors of Eqs. (2) and (7) are developed. In vibrations problems having normal modes, the expressions simplify greatly.

Analysis

A matrix solution of Eq. (1) under arbitrary initial displacements $\{x_0\}$ and velocities $\{\dot{x}_0\}$ is

$$\{x\}=[U][e^{\lambda t}][U]^{-1}\{a\}+[U^*][e^{\lambda^* t}][U^*]^{-1}\{a^*\} \tag{8}$$

in which the columns of $[U]$ are the eigenvectors $\{u_j\}$, -1 indicates the inverse, and * indicates the complex conjugate.

In order to satisfy the initial conditions,

$$\{a\}=\left[\ [U]\ [\lambda]\ [U]^{-1}-[U^*]\ [\lambda^*]\ [U^*]^{-1}\right]^{-1} \times\left\{\{\dot{x}_0\}-[U^*]\ [\lambda^*]\ [U^*]^{-1}\{x_0\}\right\} \tag{9}$$

Although $\{a\}$, $[\lambda]$, and $[U]$ are, in general, complex quantities, the motion $\{x\}$ is real. Only complex conjugate pairs of eigenvalues are considered here. The solution to Eq. (3) subject to the same initial conditions is

$$\{y\}=[V]\ [e^{\Lambda t}]\ [V]^{-1}\{y_0\} \tag{10}$$

which is identical to Eqs. (8) and (9), provided that

$$[V]=\begin{bmatrix}[U] & [U^*]\\ [U]\ [\lambda] & [U^*]\ [\lambda^*]\end{bmatrix} \tag{11}$$

$$[\Lambda]=\begin{bmatrix}[\lambda] & [0]\\ [0] & [\lambda^*]\end{bmatrix} \tag{12}$$

$$[V]^{-1}=\begin{bmatrix}-[U]^{-1}[F]^{-1}[U^*][\lambda^*][U^*]^{-1} & [U]^{-1}[F]^{-1}\\ -[U^*]^{-1}[F^*]^{-1}[U][\lambda][U]^{-1} & [U^*]^{-1}[F^*]^{-1}\end{bmatrix} \tag{13}$$

in which $[F]$ is purely imaginary:

$$[F]=[U]\ [\lambda]\ [U]^{-1}-[U^*]\ [\lambda^*]\ [U^*]^{-1} \tag{14}$$

Normal Modes

A necessary and sufficient condition to have damped normal modes is that a real transformation $[\phi]$ diagonalizes the damping as well as the mass and stiffness matrices[1]:

$$[\phi]'\ [M]\ [\phi]=[I] \tag{15}$$

$$[\phi]'\ [K]\ [\phi]=[\Omega^2] \tag{16}$$

$$[\phi]'\ [C]\ [\phi]=2[\zeta]\ [\Omega] \tag{17}$$

in which $(\ )'$ represents the transpose, $[I]$ is the identity matrix, $[\Omega]$ is the matrix of undamped natural frequencies, and $[\zeta]$ is the matrix of modal damping ratios.

Thus, the general form of this damping satisfies

$$[C]=2[M]\ [\phi]\ [\zeta]\ [\Omega]\ [\phi]^{-1} \tag{18}$$

One example of this is dyadic damping,[2] in which the modal damping ratios are given as

$$[\zeta]=[\mu] \tag{19}$$

A special case of this, for which all the damping ratios are the same, γ, has

$$[\zeta]=\gamma[I] \tag{20}$$

A generalized Rayleigh formulation[3]

$$[\zeta]=\sum_{\ell=1}^{m}\ \eta_\ell[\Omega]^{2\ell-3} \qquad m\leq n \tag{21}$$

has the so-called Rayleigh (or proportional) damping as a special case ($m=2$)[4]

$$[C]=\alpha[M]+\beta[K[\tag{22}$$

for which

$$\zeta_j=\tfrac{1}{2}(\alpha/\Omega_j+\beta\Omega_j) \tag{23}$$

Any linear combination of viscous C_v or hysteretic C_H damping satisfying Eq. (18) yields normal modes

$$C=C_v+C_H/\omega \tag{24}$$

in which ω is the frequency of vibration.[5]

For no damping, the eigenvalues of Eq. (2) are

$$\lambda_j=i\ \Omega_j \tag{25}$$

in which $i=\sqrt{-1}$ is the imaginary symbol; for damping matrices satisfying Eq. (18) the results are

$$\lambda_j=\delta_j+i\ \omega_j \tag{26}$$

in which

$$\delta_j=-\zeta_j\Omega_j \tag{27}$$

$$\omega_j=\Omega_j\sqrt{1-\zeta_j^2} \tag{28}$$

For normal modes, the following relation holds:

$$[U]=[U^*]=[\phi] \tag{29}$$

and, from Eqs. (11-13)

$$[V]=\begin{bmatrix}[\phi] & [\phi] \\ [\phi]\,[\lambda] & [\phi]\,[\lambda^*]\end{bmatrix} \tag{30}$$

$$[V]^{-1}=\frac{i}{2}\begin{bmatrix}[\omega]^{-1}[\lambda^*][\phi]'[M] & -[\omega]^{-1}[\phi]'[M] \\ -[\omega]^{-1}[\lambda]\,[\phi]'[M] & [\omega]^{-1}[\phi]'\,[M]\end{bmatrix} \tag{31}$$

in which $[\omega]$ is the imaginary part of $[\lambda]$.

With these relations, the eigenvalues and eigenvectors of one formulation readily yield those of the other. The formulas developed may be altered readily to include real as well as complex or imaginary eigenvalues by means of additional partitioning of the matrices.

Acknowledgments

The author thanks Gary C. Hart, Richard B. Nelson, and Monir Massoud for stimulating discussions that led to this Note. The National Research Council of Canada funded the research.

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Three-Dimensional Compressible Stagnation Point Boundary Layers for Laser Heated Flows

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Introduction

IN recent years, there have been several studies dealing with applications of lasers in propulsion,[1] hypersonic wind tunnels,[2] and nozzle flows.[3-4] Since the freestream temperature for a laser heated rocket thruster (with pure hydrogen as the flow medium) is very high (about 14,000 K), the accurate determination of the heat-transfer rate to the wall is of considerable importance. In order to estimate this heat-transfer rate, Wu[5] has obtained the similarity solutions for the two-dimensional and axisymmetric steady laminar stagnation-point flows with variable transport properties of hydrogen. Since, in general, the shape of a rocket thruster is three-dimensional, the solution of a three-dimensional stagnation-point flow would give more accurate estimation of the heat-transfer rate at the wall as compared to the solutions of two-dimensional or axisymmetric stagnation-point flows.

Hence, in this Note, we have obtained the similarity solution of the steady laminar compressible three-dimensional stagnation-point flow with variable transport properties of hydrogen using the quasilinearization technique.[6] The foregoing problem, with air as a flow medium, has been studied previously.[6-8] It is well known that at high temperatures the transport properties of pure hydrogen vary significantly from those of air.[9-10] In addition to the applications mentioned earlier, the present analysis also is useful in the design of space vehicles entering the atmospheres of Jupiter, Saturn, Uranus, and Neptune, where the atmosphere predominantly consists of hydrogen.[9]

Governing Equations

We consider the fluid to be pure hydrogen at a temperature of 14,000 K and at a pressure of 3 atm.[5] As in Ref. 5, we also assume that $\rho \propto h^{-1}$ and $Pr=$const (where h and Pr are the enthalpy and Prandtl number, respectively). The boundary-layer equations in dimensionless form for the steady laminar compressible three-dimensional forward stagnation-point flow, with variable transport properties under similarity assumptions, can be expressed as[6,7]

$$(\phi f'')'+(f+cs)f''+g-f'^2=0 \tag{1}$$

$$(\phi s'')'+(f+cs)s''+c(g-s'^2)=0 \tag{2}$$

$$(\phi g')'+Pr(f+cs)g'=0 \tag{3}$$

with boundary conditions

$$f(0)=f'(0)=s(0)=s'(0)=0 \quad g(0)=g_w$$

$$f'(\infty)=s'(\infty)=g(\infty)=1 \tag{4}$$

where f and s are the dimensionless stream functions; g is the dimensionless enthalpy; c is the ratio of velocity gradients at the stagnation point in x and y directions; $\phi=\rho\mu/\rho_e\mu_e$ is the ratio of the product of density and viscosity; g_w is the dimensionless enthalpy at the wall; prime denotes differentiation with respect to the independent variable η; and subscripts w and e denote conditions at the wall and at the edge of the boundary layer, respectively (other symbols are given in Ref. 6). It may be noted that most shapes of practical interest range from a sphere ($c=1$) to a cylinder[6] ($c=0$).

The skin-friction coefficients C_f and $\bar{C}_f$ along x and y directions, and the heat-transfer coefficient in terms of the Stanton number St, are given by[6-8]

$$C_f=2(Re_x)^{-1/2}F''_w \quad \bar{C}_f=2(Re_x)^{-1/2}(v_e/u_e)S''_w \tag{5}$$

$$St=(Re_x)^{-1/2}G'_w \quad F''_w=\phi_w f''_w \quad S''_w=\phi_w s''_w \tag{6}$$

$$G'_w=Pr^{-1}\phi_w g'_w/(1-g_w) \quad Re_x=u_e x/\nu_e \tag{7}$$

where F''_w and S''_w are the skin-friction parameters along the x and y directions, respectively; G'_w is the heat-transfer parameter; Re_x is the local Reynolds number; and f''_w, s''_w, and g'_w are the gradients of the longitudinal and tangential velocities and enthalpy, respectively.

Results and Discussion

Equations (1-3) under the boundary conditions of Eq. (4) have been solved numerically, using the quasilinearization technique,[6] for various values of c and g_w, where $\phi=\rho\mu/\rho_e\mu_e$ has been calculated by use of the equilibrium properties of hydrogen given in Refs. 9 and 10. As in Ref. 5, for simplicity, we have taken $Pr=1$. The variation of ϕ with η for the general three-dimensional stagnation-point flows is not presented here for lack of space.‡ However, $\phi(\eta)$ for the two-dimensional and axisymmetric stagnation-point flows is given in Fig. 1 of Ref. 5. The results corresponding to variable $\rho\mu$ (ϕ

Received Jan. 24, 1977; revision received April 14, 1977.

Index categories: Boundary Layers and Convective Heat Transfer—Laminar; Lasers.

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